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Integrable deformations of the classical Heisenberg model

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Abstract. We investigate in detail the integrable deformations of the classical Heisenberg model by use of the prolongation theory of Wahlquist and Estabrook. For the first-order deformations, our results coincide with those obtained by Mikhailov and Shabat. For the higher-order deformations, there exist arbitrary N-order isotropic integrable deformations, but the anisotropic integrable deformations are not found. Finally it is pointed out that these higher-order isotropic deformed Heisenberg spin equations are equivalent to the generalised nonlinear Schrödinger equations.

1. Introduction

It is worthwhile investigating the integrable deformations of the one-dimensional classical Heisenberg model. This is because such spin equations as, for example, the Landau–Lifshitz equation, will appear if one takes into account the anisotropy of the magnet, the magnetic-dipole interaction, biquadratic interaction and so on.

Recently, Mikhailov and Shabat [1] investigated the first-order deformations using the equivalence between the Heisenberg spin equations and the nonlinear Schrödinger equations. Some higher-order deformations have also been discussed [2, 3].

It is well known that all nonlinear evolution equations which are completely integrable possess a non-Abelian prolongation structure [4, 5]. This paper is devoted to investigating the prolongation structure of the deformed Heisenberg model, so that the integrable deformations can be obtained.

2. The first-order deformations

We consider the integrable Heisenberg spin equations of the form

$$\boldsymbol{s}_t = \boldsymbol{s} \times \boldsymbol{s}_{xx} + \boldsymbol{E}(\boldsymbol{s}, \boldsymbol{s}_x) \tag{1}$$

where $\boldsymbol{s}\cdot\boldsymbol{s} = 1$ and $\boldsymbol{s}\cdot\boldsymbol{E} = 0$. Taking $s_{a,x}$ $(a = 1, 2, 3)$ as new independent variables, then a set of 2-forms giving rise to equation (1) is

$$\begin{aligned} \alpha_a &= \mathrm{d}s_a \wedge \mathrm{d}t - s_{a,x}\, \mathrm{d}x \wedge \mathrm{d}t \\ \alpha_{3+a} &= \mathrm{d}s_a \wedge \mathrm{d}x + \varepsilon_{abc} s_b\, \mathrm{d}s_{c,x} \wedge \mathrm{d}t + E_a\, \mathrm{d}x \wedge \mathrm{d}t \\ \alpha_7 &= s_a\, \mathrm{d}s_{a,x} \wedge \mathrm{d}t + s_{a,x}\, \mathrm{d}s_a \wedge \mathrm{d}t \equiv 0 \end{aligned} \tag{2}$$

which obviously constitutes a closed ideal I, i.e.

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$$\mathrm{d}\alpha_i = \sum_{i,j} f_{ij} \wedge \alpha_j \qquad i, j = 1, 2, \ldots, 7 \tag{3}$$

where f_{ij} is some set of 1-forms. According to the prolongation scheme [4], we seek a set of 1-forms

$$\omega_k = \mathrm{d}y^k + F^k\,\mathrm{d}x + G^k\,\mathrm{d}t \qquad k = 1, 2, \ldots, n \tag{4}$$

where F^k and G^k are functions of $(s_a,\ s_{a,x};\ y^k)$ which demand that the prolonged ideal $\tilde{I} = \{\alpha_j,\ \omega_k\}$ be closed. That is

$$\mathrm{d}\omega_k = \sum_i g_i^k \alpha_i + \sum_{l=1}^{n} n_l^k \wedge \omega_l \tag{5}$$

where g_i^k and n_l^k are some sets of 0-forms and 1-forms, respectively. This requirement gives the integrabilitiy conditions of (1)

$$F^k_{s_{a,x}} = 0 \qquad G^k_{s_{a,x}} = -\varepsilon_{abc} s_b F^k_{s_c} \qquad s_{a,x} G^k_{s_a} - E_a F^k_{s_a} - [F, G]^k = 0 \tag{6}$$

where $[F, G]^k = \Sigma_{l=1}^n (F^l G^k_{yl} - G^l F^k_{yl})$.

Equations (6) have the solutions

$$F = \sum_{a=1}^{3} d_a s_a x_a \qquad G = \sum_{a=1}^{3} [d_a \nu_a + \eta(s_b) d_a s_a + h_a s_a] x_a \tag{7}$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3) = \boldsymbol{s} \times \boldsymbol{s}_x$, x_a depend only on the prolongation variables y^k and form the su(2) Lie algebra, i.e. $[x_a, x_b] = \varepsilon_{abc} x_c$ and

$$\begin{aligned}
&(d_2^2 - d_3^2)d_1 + \alpha(J_2 - J_3)d_2 d_3 + \beta(B_2 - B_3)d_1 = 0\\
&(d_3^2 - d_1^2)d_2 + \alpha(J_3 - J_1)d_1 d_3 + \beta(B_3 - B_1)d_2 = 0\\
&(d_1^2 - d_2^2)d_3 + \alpha(J_1 - J_2)d_1 d_2 + \beta(B_1 - B_2)d_3 = 0\\
&h_1 = -d_2 d_3 + \alpha J_1 d_1 \qquad h_2 = -d_1 d_3 + \alpha J_2 d_2\\
&h_3 = -d_1 d_2 + \alpha J_3 d_3\\
&\eta(s_b) = \gamma - \tfrac{1}{2}\alpha \boldsymbol{s}\cdot J\boldsymbol{s}\\
&\boldsymbol{E} = \eta(s_b)\boldsymbol{s}_x + \alpha J\boldsymbol{s}_x + s_{a,x}\eta_{s_a}\boldsymbol{s} + \beta \boldsymbol{s} \times B\boldsymbol{s}
\end{aligned} \tag{8}$$

here, $J = \mathrm{diag}(J_1, J_2, J_3)$, $B = \mathrm{diag}(B_1, B_2, B_3)$, α, β, γ, J_a and B_a are constants. In determining $\eta(s_b)$, we use the condition $\boldsymbol{s}\cdot\boldsymbol{E} = 0$.

From (8), we can see that (1) can be only parameterised in the following cases.

(i) When $\alpha = \beta = 0$, then $d_a = \delta_a \lambda$. Here $\delta_a = \pm 1$ and λ is the spectral parameter, which guarantees the integrability of (1).

(ii) When $\alpha = 0$, then $d_1 = (\rho/\mathrm{sn}(\lambda, \xi))\delta_1$, $d_2 = (\rho\,\mathrm{cn}(\lambda, \xi)/\mathrm{sn}(\lambda, \xi))\delta_2$, $d_3 = (\rho\,\mathrm{dn}(\lambda, \xi)/\mathrm{sn}(\lambda, \xi))\delta_3$, where $\rho^2 = \beta(B_1 - B_2)$, $\rho^2\xi^2 = \beta(B_1 - B_3)$.

(iii) When $\beta = 0$, then $d_1 = (\rho^2\,\mathrm{cn}(\lambda, \xi)\,\mathrm{dn}(\lambda, \xi)/\mathrm{sn}^2(\lambda, \xi))\delta_1$, $d_2 = (\rho^2\,\mathrm{cn}(\lambda, \xi)/\mathrm{sn}^2(\lambda, \xi))\delta_2$, $d_3 = (\rho^2\,\mathrm{dn}(\lambda, \xi)/\mathrm{sn}^2(\lambda, \xi))\delta_3$, where $\rho^2 = \alpha(J_1 - J_3)\delta_1\delta_2\delta_3$, $\rho^2\xi^2 = \alpha(J_1 - J_2)\delta_1\delta_2\delta_3$.

(iv) When $J_a = J_b$ and $B_a = B_b$, then $d_a = \delta_a \rho \sinh\lambda$, $d_b = \delta_b \rho \sinh\lambda$, $d_c = \delta_c \rho \cosh\lambda + \frac{1}{2}\alpha(J_a - J_c)\delta_a\delta_b$, where $\rho^2 = \frac{1}{4}\alpha^2(J_a - J_c)^2 + \beta(B_a - B_c)$.

The Lax pair for (1) is

$$\begin{aligned}
U &= F|_{x_a = -\frac{1}{2}\mathrm{i}\sigma_a} = -\tfrac{1}{2}\mathrm{i} \sum_a d_a s_a \sigma_a\\
V &= G|_{x_a = -\frac{1}{2}\mathrm{i}\sigma_a} = -\tfrac{1}{2}\mathrm{i} \sum_a (d_a \nu_a + \eta(s_b) d_a s_a + h_a s_a)\sigma_a
\end{aligned} \tag{9}$$

where σ_a are Pauli matrices.

3. The higher-order deformations

In this section, we consider the more general integrable spin equations of the form

$$\boldsymbol{s}_t = \boldsymbol{s}\times\boldsymbol{s}_{xx} + \varepsilon_N \boldsymbol{E}_N(\boldsymbol{s}, \boldsymbol{s}_x, \ldots, \boldsymbol{s}_{(N-1)x}) + \begin{cases} (-1)^{(N-1)/2}\varepsilon_N[\boldsymbol{s}_{Nx} - (\boldsymbol{s}\cdot\boldsymbol{s}_{Nx})\boldsymbol{s}] & \text{if } N \text{ odd} \\ (-1)^{N/2}\varepsilon_N \boldsymbol{s}\times\boldsymbol{s}_{Nx} & \text{if } N \text{ even} \end{cases} \tag{10}$$

where $\boldsymbol{s}_{px} = \partial^p \boldsymbol{s}/\partial x^p$, N is an arbitrary positive integer and ε_N is the deformation parameter. When $N=1$ and 2, (10) reduces the first-order deformed spin equations discussed in section 2. Here, we restrict $N \geqslant 3$.

Performing the same procedures in section 2, we obtain the integrability conditions for (10) as follows:

$$F^k_{s_{a,ix}} = 0$$

$$G^k_{s_{a,(N-1)x}} = \begin{cases} (-1)^{(N-1)/2}\varepsilon_N(F^k_{s_a} - s_b F^k_{s_b} s_a) & \text{if } N \text{ odd} \\ (-1)^{(N/2)-1}\varepsilon_N \varepsilon_{abc} s_b F^k_{s_c} & \text{if } N \text{ even} \end{cases} \qquad i = 1, 2, \ldots, N-1$$

$$s_{a,ix} G^k_{s_{a,(i-1)x}} + \varepsilon_{abc} s_{a,2x} s_b F^k_{s_c} - \varepsilon_N E_a F^k_{s_a} - [F, G]^k = 0. \tag{11}$$

We assume that the equations above have the solutions

$$F = \lambda s_a x_a = \lambda S \qquad G = \lambda[S, S_x] - \lambda^2 S + \varepsilon_N \sum_{p=1}^{N} \lambda^P \Theta_{N-p+1} \tag{12}$$

where $\Theta_r = \Theta^a_r(s_a, s_{a,x}, \ldots, s_{a,(r-1)x}) x_a$.

Substituting (12) into (11), we have

$$[S, \Theta_1] = 0 \tag{13a}$$

$$s_{a,ix}\Theta_{r,s_{a,(i-1)x}} = [S, \Theta_{r+1}] \qquad r = 1, 2, \ldots, N-1 \tag{13b}$$

$$E_N = s_{a,ix}\Theta_{N,s_{a,(i-1)x}}. \tag{13c}$$

From (13a), we have $\Theta_1 = S$. Using the (13b) and (13c) and the condition $s_b s_{a,ix}\Theta^b_{r,s_{a,(i-1)x}} = 0$, we can obtain Θ_r and $\boldsymbol{E}_N$. We note that the condition $\boldsymbol{s}\cdot\boldsymbol{E} = 0$ is self-satisfied. Now, we write down the several higher-order isotropic integrable deformed spin equations.

(i) When $N=3$, we have

$$\boldsymbol{s}_t = \boldsymbol{s}\times\boldsymbol{s}_{xx} - \tfrac{3}{2}\varepsilon_3(\boldsymbol{s}_x\cdot\boldsymbol{s}_x)\boldsymbol{s}_x - 3\varepsilon_3(\boldsymbol{s}_x\cdot\boldsymbol{s}_{xx})\boldsymbol{s} - \varepsilon_3\boldsymbol{s}_{3x}. \tag{14}$$

(ii) When $N=4$, we have

$$\boldsymbol{s}_t = \boldsymbol{s}\times\boldsymbol{s}_{xx} + \tfrac{5}{2}\varepsilon_4(\boldsymbol{s}_x\cdot\boldsymbol{s}_x)\boldsymbol{s}\times\boldsymbol{s}_{xx} + 5\varepsilon_4(\boldsymbol{s}_x\cdot\boldsymbol{s}_{xx})\boldsymbol{s}\times\boldsymbol{s}_x + \varepsilon_4\boldsymbol{s}\times\boldsymbol{s}_{4x}. \tag{15}$$

(iii) When $N=5$, we have

$$\begin{aligned}\boldsymbol{s}_t = \boldsymbol{s}\times\boldsymbol{s}_{xx} &+ \varepsilon_5[\tfrac{35}{2}(\boldsymbol{s}_x\cdot\boldsymbol{s}_x)(\boldsymbol{s}_x\cdot\boldsymbol{s}_{xx}) + 10\boldsymbol{s}_{xx}\,\boldsymbol{s}_{3x} + 5\boldsymbol{s}_x\cdot\boldsymbol{s}_{4x}]\boldsymbol{s} \\ &+ \varepsilon_5[\tfrac{35}{8}(\boldsymbol{s}_x\cdot\boldsymbol{s}_x)^2 + 10\boldsymbol{s}_x\cdot\boldsymbol{s}_{3x} + \tfrac{15}{2}\boldsymbol{s}_{xx}\cdot\boldsymbol{s}_{xx}]\boldsymbol{s}_x + 10\varepsilon_5(\boldsymbol{x}_x\cdot\boldsymbol{s}_{xx})\boldsymbol{s}_{xx} \\ &+ \tfrac{5}{2}\varepsilon_5(\boldsymbol{s}_x\cdot\boldsymbol{s}_x)\boldsymbol{s}_{3x} + \varepsilon_5\boldsymbol{s}_{5x}.\end{aligned} \tag{16}$$

4. Equivalence

The equivalence between (14) and (15) and the generalised nonlinear Schrödinger equations has been discussed in [2, 3]. In this section, we shall point out that (16) is also equivalent to the generalised nonlinear Schrödinger equation, which is also completely integrable. It is obvious that this equivalence exists for the much higher-order isotropic deformed spin equations.

Following Lakshmanan [6], we map (16) on a moving helical space curve described by the orthogonal trihedral $\boldsymbol{e}_a$ which satisfy the Serret-Frenet equations

$$\boldsymbol{e}_{1x} = k\boldsymbol{e}_2 \qquad \boldsymbol{e}_{2x} = -k\boldsymbol{e}_1 + \tau\boldsymbol{e}_3 \qquad \boldsymbol{e}_{3x} = -\tau\boldsymbol{e}_2 \tag{17}$$

where the curvature is given by $k = (\boldsymbol{e}_{1x}\cdot\boldsymbol{e}_{1x})^{1/2}$ and the torsion is given by $\tau = k^{-2}\boldsymbol{e}_1\cdot(\boldsymbol{e}_{1x}\times\boldsymbol{e}_{1xx})$. Taking $\boldsymbol{e}_1 = \boldsymbol{s}$ and using (16) and (17), we obtain

$$\boldsymbol{e}_{at} = (\Omega_b\boldsymbol{e}_b)\times\boldsymbol{e}_a \tag{18}$$

where

$$\begin{aligned}
\Omega_1 &= k^{-1}k_{xx} - \tau^2 + \varepsilon_5(\tfrac{3}{8}k^4\tau - 3k^2\tau^3 + \tau^5 + \tfrac{17}{2}kk_{xx}\tau + \tfrac{29}{2}k_x^2\tau + 5k^{-1}k_{4x}\tau \\
&\quad - 10k^{-1}k_{xx}\tau^3 - 30k^{-1}k_x\tau^2\tau_x - 10\tau^2\tau_{xx} - 15\tau\tau_x^2 + \tfrac{21}{2}kk_x\tau_x \\
&\quad + \tfrac{3}{2}k^2\tau_{xx} + 10k^{-1}k_{3x}\tau_x + 10k^{-1}k_{xx}\tau_{xx} + 5k^{-1}k_x\tau_{3x} + \tau_{4x}) \\
\Omega_2 &= -k_x + \varepsilon_5(-6k^2k_x\tau - \tfrac{3}{2}k^3\tau_x - 4k_{3x}\tau + 4k_x\tau^3 + 6k\tau^2\tau_x \\
&\quad - 6k_{xx}\tau_x - 4k_x\tau_{xx} - k\tau_{3x}) \\
\Omega_3 &= -k\tau + \varepsilon_5(\tfrac{3}{8}k^5 + k\tau^4 - 3k^3\tau^2 + \tfrac{5}{2}k^2k_{xx} + \tfrac{5}{2}kk_x^2 + k_{4x} \\
&\quad - 6k_{xx}\tau^2 - 12k_x\tau\tau_x - 4k\tau\tau_{xx} - 3k\tau_x^2).
\end{aligned} \tag{19}$$

Using the compatibility condition $\boldsymbol{e}_{a,xt} = \boldsymbol{e}_{a,tx}$, we have

$$\begin{aligned}
k_t &= -2k_x\tau - k\tau_x + \varepsilon_5(-15k^2k_x\tau^2 - \tfrac{15}{2}k^3\tau\tau_x - 10k_{3x}\tau^2 + 5k_x\tau^4 + 10k\tau^3\tau_x \\
&\quad - 30k_{xx}\tau\tau_x - 20k_x\tau\tau_{xx} - 5k\tau\tau_{3x} + \tfrac{15}{8}k^4k_x + 10kk_xk_{xx} \\
&\quad + \tfrac{5}{2}k^2k_{3x} + \tfrac{5}{2}k_x^3 - 15k_x\tau_x^2 - 10k\tau_x\tau_{xx} + k_{5x}) \\
\tau_t &= [\tfrac{1}{2}k^2 - \tau^2 + k^{-1}k_{xx} + \varepsilon_5(\tfrac{15}{8}k^4\tau - 5k^2\tau^3 + \tfrac{25}{2}kk_{xx}\tau + \tfrac{25}{2}k_x^2\tau - 10\tau^2\tau_{xx} \\
&\quad + \tau^5 - 15\tau\tau_x^2 + \tfrac{25}{2}kk_x\tau_x + \tfrac{5}{2}k^2\tau_{xx} + \tau_{4x} + 5k^{-1}k_{4x}\tau \\
&\quad - 10k^{-1}k_{xx}\tau^3 - 30k^{-1}k_x\tau^2\tau_x + 10k^{-1}k_{3x}\tau_x \\
&\quad + 10k^{-1}k_{xx}\tau_{xx} + 5k^{-1}k_x\tau_{3x})]_x.
\end{aligned} \tag{20}$$

Taking the complex transformation [6]

$$\psi(x, t) = 2k(x, t)\exp\left(\mathrm{i}\int_{-\infty}^{x}\tau(y, t)\,\mathrm{d}y\right) \tag{21}$$

equations (20) then become the generalised nonlinear Schrödinger equation

$$\mathrm{i}\psi_t + \psi_{xx} + 2|\psi|^2\psi - \mathrm{i}\varepsilon_5[10(|\psi_x|^2\psi)_x + 30|\psi|^4\psi_x + 20\psi^*\psi_x\psi_{xx} + 10|\psi|^2\psi_{3x} + \psi_{5x}] = 0 \tag{22}$$

which is the equivalent form of (16).

The Lax pair associated with (22) is

$$
\begin{aligned}
U_5 &= -\tfrac{1}{2}\mathrm{i}\lambda\sigma_3 + \tfrac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2)\psi - \tfrac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2)\psi^* \\
V_5 &= A\sigma_3 + \tfrac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2)B - \tfrac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2)B^*
\end{aligned}
\qquad (23)
$$

where

$$
\begin{aligned}
A = {} & \mathrm{i}|\psi|^2 + \tfrac{1}{2}\mathrm{i}\lambda^2 + \varepsilon_5(\psi^*\psi_{3x} - \psi\psi^*_{3x} + \psi_x\psi^*_{xx} - \psi_{xx}\psi^*_x + 6|\psi|^2\psi^*\psi_x - 6|\psi|^2\psi\psi^*_x) \\
& - \mathrm{i}\lambda\varepsilon_5(\psi\psi^*_{xx} + \psi^*\psi_{xx} - |\psi_x|^2 + 3|\psi|^4) \\
& + \lambda^2\varepsilon_5(\psi\psi^*_x - \psi^*\psi_x) + \mathrm{i}\lambda^3\varepsilon_5|\psi|^2 \qquad (24) \\
B = {} & \lambda\psi + \varepsilon_5(\psi_{4x} + 8|\psi|^2\psi_{xx} + 2\psi^2\psi^*_{xx} + 4|\psi_x|^2\psi + 6\psi_x^2\psi^* + 6|\psi|^4\psi) + \mathrm{i}\psi_x \\
& - \mathrm{i}\lambda\varepsilon_5(\psi_{3x} + 6|\psi|^2\psi_x) - \lambda^2\varepsilon_5(\psi_{xx} + 2|\psi|^2\psi) + \mathrm{i}\lambda^3\varepsilon_5\psi_x + \lambda^4\varepsilon_5\psi.
\end{aligned}
$$

One can verify directly that (16) and (22) are also gauge equivalent under the transformations [7]

$$
\Phi = g^{-1}\phi \qquad U_5 = g^{-1}u_5 g - g^{-1}g_x \qquad V_5 = g^{-1}v_5 g - g^{-1}g_t \qquad (25)
$$

where U_5 and V_5 denote the Lax pair for (16), $u_5|_{\lambda=0} = g_x g^{-1}$, $v_5|_{\lambda=0} = g_t g^{-1}$ and $s_a\sigma_a = g^{-1}\sigma_3 g$.

5. Conclusion

We have investigated the integrable deformations of the Heisenberg model. From (11), we do not obtain the anisotropic integrable deformations. It is obvious that the linear combinations of these isotropic deformations are also completely integrable, i.e.

$$
\boldsymbol{s}_t = \boldsymbol{s} \times \boldsymbol{s}_{xx} + \sum_{j=1}^{N} \varepsilon_j \left(\boldsymbol{E}_j + \begin{cases} (-1)^{(j-1)/2}[\boldsymbol{s}_{jx} - (\boldsymbol{s}\cdot\boldsymbol{s}_{jx})\boldsymbol{s}] & \text{if } j \text{ odd} \\ (-1)^{j/2}\boldsymbol{s} \times \boldsymbol{s}_{jx} & \text{if } j \text{ even} \end{cases} \right) \qquad (26)
$$

are integrable. This property can be seen easily from the integrability conditions of (26). The equations above are also equivalent to the generalised nonlinear Schrödinger equations.

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